

Entropy of Kerr-Newman Black Hole to All Orders in the Planck Length

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Abstract Using the quantum statistical method, the difficulty of solving the wave equation on the background of the black hole is avoided. We directly solve the partition functions of Bose and Fermi field on the background of an axisymmetric Kerr-Newman black hole using the new equation of state density motivated by the generalized uncertainty principle in the quantum gravity. Then near the black hole horizon, we calculate entropies of Bose and Fermi field between the black hole horizon surface and the hypersurface with the same inherent radiation temperature measured by an observer at an infinite distance. In our results there are not cutoffs and little mass approximation introduced in the conventional brick-wall method. The series expansion of the black hole entropy is obtained. And this series is convergent. It provides a way for studying the quantum statistical entropy of a black hole in a non-spherical symmetric spacetime.

Keywords Generalized uncertainty principle · Black hole entropy · Non-spherical symmetric spacetime

1 Introduction

In the early 1970s, Bekenstein, Hawking, Bardeen et al. proposed that the black hole entropy is proportional to the area of event horizon [1–3]. Hereafter research on black hole thermodynamics makes rapid progress. Especially the proof of Hawking radiation has been effective in stimulating the enthusiasm of scientists for black hole thermal property [2]. If the gravity at a black hole's surface κ is taken as temperature and the area of horizon A is taken as the entropy, four laws of thermodynamics in black hole's theory can be derived. Hereafter, black hole thermodynamics has received considerable attention. Especially researchers pay widespread attention to the statistical origin of the black hole entropy. Many methods of calculating entropy have emerged [4–9]. One of them is the well-known brick-wall method [7]. Statistical properties of free scalar fields on background of various black

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holes are discussed by this method [10–14]. The entropy expression with respect to the horizon area is derived. It is shown that the entropy is directly proportional to its outer horizon area. For Schwarzschild spacetime [7], when cutoffs satisfy a proper condition, the entropy can be written as $S = A_H/4$. When cutoffs approach zero, the entropy diverges. 't Hooft thought that this divergence was caused by the fact the state density would approach infinite near the horizon. Subsequently, it is found that the quantum states make a leading contribution to the black hole entropy near the horizon. Thus, the brick-wall model has been improved and the thin layer model has been proposed [15–17]. The thin layer model only considers the quantum states in a thin layer near the horizon. The infrared cutoff and little mass approximation in the original brick-wall model are avoided. However, there still exists ultraviolet cutoff. Recently, it is found that the generalized uncertainty principle is related to state density [18–25]. The statistical entropies have been calculated using the generalized uncertainty principle [26–32]. The simplest way to generalize the uncertainty principle is to promote it to [30]

$$\Delta x \Delta p \geq \frac{1}{2} (1 + \lambda (\Delta p)^2) \quad (1)$$

and the correction to state density is

$$dn = \frac{d^3 x d^3 p}{(2\pi)^3 (1 + \lambda p^2)^3}. \quad (2)$$

The statistical entropy of the scalar field on the background of Reissner-Nordstrom black hole was calculated and the series expansion of the entropy was derived near the black hole horizon. Note that the higher-order terms of the entropy are divergent [30]. Reference [32] also discussed the statistical entropy of the scalar field on the background of Reissner-Nordstrom black hole and the series expansion of the entropy was derived near the black hole horizon. Recently, [20–22] proposed the correction to state density due to the generalized uncertainty principle is as follows:

$$dn = \frac{d^3 x d^3 p}{(2\pi)^3} e^{-\lambda p^2}, \quad (3)$$

where $p^2 = p^i p_i$, λ is a constant characterized the correction to Heisenberg uncertainty principle from the gravitation. λ is the same with Planck area quantitatively.

In this paper, we calculate the statistical entropy of an axisymmetric Kerr-Newman black hole by using the correction relation (3) to state density due to the generalized uncertainty principle. Since Kerr-Newman spacetime is axisymmetric, in calculation, the integral interval is taken from the black hole horizon surface to the hypersurface with the same inherent radiation temperature measured by an observer at an infinite distance. The inherent thickness of two surfaces are the same with the minimal length quantitatively introduced in the generalized uncertainty principle. And this thin layer clings to the black hole horizon. Under the case without any artificial cutoff and little mass approximation, we derive the series expansions of the statistical entropies of Bose and Fermi field. And the leading terms of these expansions are directly proportional to the area of the black hole horizon. Farther, this series is convergent that the black hole entropy can be expressed as a convergence series. There does not exist any artificial cutoff and little mass approximation. It is shown that there is inherent relation between the black hole entropy and the horizon area. Therefore, it makes people to have a better understanding of the black hole statistical entropy in non-spherical symmetry spacetimes. Because we adopt the quantum statistical method, the difficulty of

solving the wave equation in the conventional brick-wall method is avoided. In the whole process, the calculation is simple, the result is reasonable. We provide a method for studying the quantum statistical entropy of a black hole in a non-spherical symmetric spacetime. We take the simplest function form of temperature ($c = \hbar = G = K_B = 1$).

2 Entropy of Bose Field

Linear element of Kerr-Newman spacetime:

$$\begin{aligned}
 ds^2 = & - \left(1 - \frac{2Mr - Q^2}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\
 & + \left[(r^2 + a^2) \sin^2 \theta + \frac{(2Mr - Q^2)a^2 \sin^4 \theta}{\rho^2} \right] d\varphi^2 \\
 & - \frac{2(2Mr - Q^2)a \sin^2 \theta}{\rho^2} dt d\varphi,
 \end{aligned} \tag{4}$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2 + Q^2$.

The radiation temperature of the black hole is as follows:

$$T_+ = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \tag{5}$$

where $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$ are the locations of outer and inner horizons of the black hole respectively. The area of the black hole horizon is

$$A(r_+) = 4\pi(r_+^2 + a^2). \tag{6}$$

The natural radiation temperature got by the observer at rest at an infinite distance is as follows [33]:

$$T = \frac{T_+}{\sqrt{-g'_{tt}}} \tag{7}$$

where T_+ is the equilibrium temperature.

$$g'_{tt} = \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = - \frac{(r - r_+)(r - r_-)(r^2 + a^2 \cos^2 \theta)}{(r^2 + a^2)^2 - (r - r_+)(r - r_-)a^2 \sin^2 \theta}. \tag{8}$$

For Bose gas, the partition function Z satisfies:

$$\ln Z = - \sum_i g_i \ln(1 - e^{-\beta \epsilon_i}). \tag{9}$$

For spacetime (4), the area of two-dimensional curved surface at arbitrary point r outside horizon is

$$A(r) = \int dA = \int \sqrt{g} d\theta d\varphi, \tag{10}$$

where $g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta} g_{\varphi\varphi}$. Then the volume of the thin layer at arbitrary point r outside the horizon is as follows:

$$dV = A(r)\sqrt{g_{rr}}dr. \tag{11}$$

So, the partition function of the system at the thin layer with arbitrary thickness at point r outside the horizon is as follows:

$$\begin{aligned} \ln Z &= - \int A(r)\sqrt{g_{rr}}dr \sum_i g_i \ln(1 - e^{-\beta\epsilon_i}) \\ &= - \int \frac{A(r)\sqrt{g_{rr}}dr}{2\pi^2} \int_0^\infty p^2 dp e^{-\lambda p^2} \ln(1 - e^{-\beta\omega_0}) \\ &\approx \int A(r)\sqrt{g_{rr}}dr \int_{m\sqrt{-g'_{tt}}}^\infty \frac{\beta_0}{6\pi^2(e^{\beta\omega_0} - 1)} p^3 e^{-\lambda p^2} d\omega, \end{aligned} \tag{12}$$

where $\beta = \beta_0\sqrt{-g'_{tt}}$. The free energy of the system is

$$F = -\frac{1}{\beta_0} \ln Z = - \int A(r)\sqrt{g_{rr}}dr \int_{m\sqrt{-g'_{tt}}}^\infty \frac{1}{6\pi^2(e^{\beta\omega_0} - 1)} p^3 e^{-\lambda p^2} d\omega. \tag{13}$$

The entropy of the system is

$$\begin{aligned} S_B &= \beta_0^2 \frac{\partial F}{\partial \beta_0} = \beta_0^2 \int A(r)\sqrt{g_{rr}}dr \int_{m\sqrt{-g'_{tt}}}^\infty \frac{\omega e^{\beta\omega_0}}{6\pi^2(e^{\beta\omega_0} - 1)^2} p^3 e^{-\lambda p^2} d\omega \\ &= \beta_0^2 \int A(r)\sqrt{g_{rr}}dr \int_{m\sqrt{-g'_{tt}}}^\infty \frac{\omega e^{\frac{\beta\omega}{\sqrt{-g'_{tt}}}}}{6\pi^2(e^{\frac{\beta\omega}{\sqrt{-g'_{tt}}} - 1})^2} e^{-\lambda(\frac{\omega^2}{-g'_{tt}} - m^2)} \left(\frac{\omega^2}{-g'_{tt}} - m^2\right)^{3/2} d\omega \\ &= \frac{1}{6\pi^2} \int A(r)\sqrt{g_{rr}}dr \int_{m\beta}^\infty \frac{x e^x}{(e^x - 1)^2} e^{-\lambda(\frac{x^2}{\beta^2} - m^2)} \left(\frac{x^2}{\beta^2} - m^2\right)^{3/2} dx. \end{aligned} \tag{14}$$

In the above calculation, we have used the relation among energy, momentum and mass, $\frac{\omega^2}{-g'_{tt}} = p^2 + m^2$, and m is a static mass of particle. In (14) the integral with respect to r is near horizon, so $g'_{tt}(r_+) \rightarrow 0$. Equation (14) can be reduced as:

$$\begin{aligned} S_B &= \frac{1}{6\pi^2} \int A(r)\sqrt{g_{rr}}dr \int_0^\infty \frac{x^4 e^x}{\beta^3(e^x - 1)^2} e^{-\lambda\frac{x^2}{\beta^2}} dx \\ &= \frac{1}{6\pi^2\beta_0^3} \int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}dr d\theta d\varphi}{(-g'_{tt})^{3/2}} \int_0^\infty \frac{x^4 e^x}{(e^x - 1)^2} e^{-\lambda\frac{x^2}{\beta^2}} dx \\ &= \frac{1}{6\pi^2\beta_0^3} \int_0^\infty \frac{dx}{4\sinh^2(\frac{x}{2})} \int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}}{(-g'_{tt})^{3/2}} x^4 e^{-\lambda\frac{x^2}{\beta^2}} dr d\theta d\varphi \\ &= \frac{1}{6\pi^2\beta_0^3} \int_0^\infty \frac{dx}{4\sinh^2(\frac{x}{2})} I_1(x, \varepsilon), \end{aligned} \tag{15}$$

where

$$\begin{aligned}
 I_1(x, \varepsilon) &= \int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}}{(-g'_{tt})^{3/2}} x^4 e^{-\lambda \frac{x^2}{\beta^2}} dr d\theta d\varphi \\
 &= \beta_0^4 \int x^4 \sqrt{-g_{\theta\theta}g_{\varphi\varphi}g_{rr}g'_{tt}} \frac{\partial^2}{\partial \lambda^2} e^{-\lambda \frac{x^2}{\beta^2}} dr d\theta d\varphi \\
 &= \beta_0^4 \int (r^2 + a^2 \cos^2 \theta) \sin \theta \frac{\partial^2}{\partial \lambda^2} e^{-\lambda \frac{x^2}{\beta^2}} dr d\theta d\varphi.
 \end{aligned} \tag{16}$$

From (7) and (8), we obtain that outside the black hole horizon at arbitrary point R ($R > r_+$) the natural radiation temperature got by the observer at rest at an infinite distance is different. It is related to angle θ . When the spacetime is spherically symmetric, outside the black hole horizon at arbitrary point R ($R > r_+$) the natural radiation temperature got by the observer at rest at an infinite distance is the same. So, to calculate the statistical entropy of quantum field near the black hole horizon, the integral interval usually is taken as $[r_+, r_+ + \varepsilon]$, and ε is a positive small quantity. From other viewpoint, the integral interval is from the black hole horizon surface to the hypersurface with the same inherent radiation temperature measured by an observer at an infinite distance. It is shown that for axisymmetric spacetime, to calculate the quantum statistical entropy near outer of the black hole horizon, the integral interval should be taken from the black hole horizon surface to the hypersurface with the same inherent radiation temperature measured by an observer at an infinite distance. From (7) and (8), near outer of the horizon, when R satisfies

$$R = r_+ + \frac{\varepsilon}{r_+^2 + a^2 \cos^2 \theta}, \tag{17}$$

an observer at rest at an infinite distance can get the hypersurface with the same inherent radiation temperature near outer of the black hole horizon. r_+ is the location of the black hole event horizon and satisfies $g'_{tt}(r_+) = 0$. ε is a positive small quantity. Near the horizon, the metric can be simply written as $g'_{tt}(r) \approx (g'_{tt})'(r_+)(r - r_+)$. From the metric (4), the minimal length is obtained as

$$\sqrt{\frac{e\lambda}{2}} = \int_{r_+}^R \sqrt{g_{rr}(r)} dr \approx \int_{r_+}^R \frac{\sqrt{r_+^2 + a^2 \cos^2 \theta}}{\sqrt{(r - r_+)(r_+ - r_-)}} dr = 2\sqrt{\frac{\varepsilon}{r_+ - r_-}}. \tag{18}$$

Near the horizon, from (32) in [34], $I_1(x, \varepsilon)$ can be expressed as:

$$\begin{aligned}
 I_1(x, \varepsilon) &= \beta_0^4 2\pi \int_{r_+}^R (r^2 + a^2 \cos^2 \theta) \sin \theta \frac{\partial^2}{\partial \lambda^2} e^{-\lambda \frac{x^2}{\beta^2}} dr d\theta \\
 &= \beta_0^4 2\pi \frac{\partial^2}{\partial \lambda^2} \int_{r_+}^R \frac{(r^2 + a^2 \cos^2 \theta) \sin \theta dr d\theta}{\sum_{n=0} \frac{1}{n!} (\frac{\lambda x^2}{-\beta_0^2 g'_{tt}(r)})^n} \\
 &\approx \beta_0^6 2\pi \frac{\partial^2}{\partial \lambda^2} \int_{r_+}^R \frac{(r_+^2 + a^2 \cos^2 \theta) \sin \theta d\theta}{\lambda x^2 - \beta_0^2 (g'_{tt})'(r_+)(r - r_+) + o((r - r_+)^2)} \\
 &\quad \times (g'_{tt})'(r_+)(r - r_+) dr.
 \end{aligned} \tag{19}$$

Neglect the higher-order terms, (19) can be rewritten as:

$$\begin{aligned}
 I_1(x, \varepsilon) &= \beta_0^4 2\pi \frac{\partial^2}{\partial \lambda^2} \int_{r_+}^R \left[1 - \frac{\lambda x^2}{\lambda x^2 - \beta_0^2 (g'_{tt})'(r_+) (r - r_+)} \right] (r_+^2 + a^2 \cos^2) \sin \theta d\theta dr \\
 &= \beta_0^4 4\pi \frac{\partial^2}{\partial \lambda^2} \left[\varepsilon + \frac{\lambda x^2 (r_+^2 + a^2)^2}{\beta_0^2 (r_+ - r_-)} \ln \left(\frac{\lambda x^2}{\lambda x^2 + \beta_0^2 \frac{(r_+ - r_-)\varepsilon}{(r_+^2 + a^2)^2}} \right) \right] \\
 &= \beta_0^6 4\pi \frac{(r_+ - r_-)}{(r_+^2 + a^2)^2} \frac{x^2 \varepsilon^2}{\lambda (\lambda x^2 + \beta_0^2 \frac{(r_+ - r_-)\varepsilon}{(r_+^2 + a^2)^2})^2} \\
 &= \frac{\beta_0^3 \pi^3 A(r_+)}{\lambda} \frac{e^2 x^2}{(x^2 + 2e\pi^2)^2}. \tag{20}
 \end{aligned}$$

Substituting (20) into (15), we obtain

$$S_B = \frac{\pi A(r_+) e^2}{48\lambda} \int_0^\infty \frac{x^2}{\sinh^2 x} \frac{dx}{(x^2 + e\pi^2/2)^2}. \tag{21}$$

Then, the integrand in (21) can be regarded as a complex function

$$f(z) = \frac{z^2}{\sinh^2 z (z^2 + e\pi^2/2)^2}. \tag{22}$$

When $n \neq 0$ are integers, $z = in\pi$ and $z = i\sqrt{\frac{e}{2}}\pi$ are two-order poles of $f(z)$. The residues are as following respectively

$$-\frac{i}{2 \sin^2(\sqrt{e/2}\pi)} \left[\operatorname{ctg}(\sqrt{e/2}\pi) - \frac{1}{2\sqrt{e/2}\pi} \right]$$

and $-2in(n^2 + e/2)/[\pi^3(n^2 - e/2)^3]$.

By the residue theorem, the entropy becomes

$$S_B = \frac{\pi A(r_+) e^2}{48\lambda} \left(\frac{\pi \operatorname{ctg}(\sqrt{e/2}\pi)}{2 \sin^2(\sqrt{e/2}\pi)} - \frac{\pi}{4\sqrt{e/2} \sin^2(\sqrt{e/2}\pi)} + \frac{2}{\pi^2} \sum_{n=1}^\infty \frac{n(n^2 + e/2)}{(n^2 - e/2)^3} \right). \tag{23}$$

If the minimal length introduced in the generalized uncertainty principle is assumed as

$$\lambda = \frac{\pi^2 e^2}{48} \frac{2\sqrt{e/2} \operatorname{ctg}(\sqrt{e/2}\pi) - 1}{\sqrt{e/2} \sin^2(\sqrt{e/2}\pi)} \tag{24}$$

in the series expansion of the entropy equation (23), the leading term is a quarter of the horizon area, which satisfies B-H entropy. Research on the correction to the black hole entropy is one of the key issues. Many methods discussing the correction to the black hole entropy have emerged [35–39], and the results are valuable. However, we still need systematically discuss it. In this paper, the correction to B-H entropy is obtained by calculating the statistical entropies of Bose and Fermi field near the horizon of the black hole. Note that the series expression of the correction is convergent, so that our result is reliable.

3 Entropy of Fermi Field

For Fermi system, the partition function is

$$\ln Z = \sum_i g_i \ln(1 + e^{-\beta \epsilon_i}). \tag{25}$$

Based on (14), we can obtain the entropy of Fermi system

$$\begin{aligned} S_F &= \frac{1}{6\pi^2} \int A(r) \sqrt{g_{rr}} dr \int_0^\infty \frac{x^4 e^x}{\beta^3 (e^x + 1)^2} e^{-\lambda \frac{x^2}{\beta^2}} dx \\ &= \frac{1}{6\pi^2 \beta_0^3} \int_0^\infty \frac{dx}{4 \cosh^2(\frac{x}{2})} \int_{r_+}^{r_+ + \epsilon} \frac{1}{g^2(r)} x^4 e^{-\lambda \frac{x^2}{\beta^2}} A(r) dr \\ &= \frac{1}{6\pi^2 \beta_0^3} \int_0^\infty \frac{dx}{4 \cosh^2(\frac{x}{2})} I(x, \epsilon) \\ &= \frac{\pi A(r_+) e^2}{48\lambda} \int_0^\infty \frac{x^2}{\cosh^2 x} \frac{dx}{(x^2 + e\pi^2/2)^2}. \end{aligned} \tag{26}$$

Therefore, we can obtain the entropy corresponding Fermi field near the black hole horizon

$$\begin{aligned} S_F &= \frac{\pi A(r_+) e^2}{48\lambda} \left(\frac{\pi t g(\sqrt{e/2}\pi)}{2 \cos^2(\sqrt{e/2}\pi)} - \frac{\pi}{4\sqrt{e/2} \cos^2(\sqrt{e/2}\pi)} \right. \\ &\quad \left. + \frac{2}{\pi^2} \sum_{n=0} \frac{(n + 1/2)((n + 1/2)^2 + e/2)}{((n + 1/2)^2 - e/2)^3} \right). \end{aligned} \tag{27}$$

If the minimal length introduced in the generalized uncertainty principle is assumed as

$$\lambda = \frac{\pi^2 e^2}{48} \frac{2\sqrt{e/2} t g(\sqrt{e/2}\pi) - 1}{\sqrt{e/2} \cos^2(\sqrt{e/2}\pi)} \tag{28}$$

the leading term in series expansion of Fermi entropy is directly proportional to the horizon area, which satisfies the Bekenstein-Hawking formula.

4 Conclusion

Taking into account the effect of the generalized uncertainty principle on the state density, we calculate the statistical entropy of the Bose and Fermi field near the black hole horizon. The series expansion of the statistical entropy is obtained without any artificial cutoff and little mass approximation. When dimensionless constant λ in the generalized uncertainty principle satisfies (24) and (28), the leading term in the series expression of the statistical entropy is proportional to the area of horizon. It satisfies the conditions of B-H entropy. In our calculation, the difficulty in solving the wave equation is overcome by the quantum statistical method. For spherically symmetric spacetime, to calculate the black hole entropy we proposed the integral interval should be: from the black hole horizon surface to the hypersurface with the same inherent radiation temperature measured by an observer at an infinite distance near outer of the black hole horizon. It also provides a way for studying the quantum statistical entropy of a black hole in a non-spherical symmetric spacetime.

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